Designing Inflatable Structures: Supplemental Material

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1 Derivatives of Principal Stretches

Let $\mathbf{C}$ denote the right Cauchy-Green tensor $\mathbf{C} = \mathbf{F}^t\mathbf{F}$, where $\mathbf{F}$ is the deformation gradient. Here we restrict considerations to the two-dimensional setting, noting that an extension to three dimensions is straightforward through kinematic assumptions [1]. We express $\mathbf{C}$ in terms of principal stretches (eigenvalues) λ_i and corresponding principal directions (eigenvectors) $\mathbf{N}_i$ as

$$\mathbf{C} = \sum_i \lambda_i \mathbf{N}_i \mathbf{N}_i^t \ , \quad \text{where} \quad \lambda_i = \mathbf{N}_i^t \mathbf{C} \mathbf{N}_i \ . \tag{1}$$

Without loss of generality, we assume that $\mathbf{N}_1$ corresponds to the direction of maximum stretch λ_1. In our two-dimensional setting, the unit-length vector $\mathbf{N}_1$ can be parametrized by a single scalar α as

$$\mathbf{N}_1(\alpha) = (\cos(\alpha), \sin(\alpha))^t \tag{2}$$

The second eigenvector $\mathbf{N}_2$ of $\mathbf{C}$ is orthogonal to $\mathbf{N}_1$

$$\mathbf{N}_2(\alpha) = (-\sin(\alpha), \cos(\alpha))^t \ . \tag{3}$$

We note that

$$\mathbf{N}_2 = \frac{\partial \mathbf{N}_1}{\partial \alpha} \quad \text{and} \quad \mathbf{N}_1 = -\frac{\partial \mathbf{N}_2}{\partial \alpha} \ . \tag{4}$$

For computing the derivatives of the principal stretches, we will focus on λ_1 for the sake of conciseness. Analogous expressions hold for λ_2. The first derivative of the maximum stretch is determined as

$$\begin{aligned}
\frac{\partial \lambda_1}{\partial \mathbf{x}} &= \frac{\partial}{\partial \mathbf{x}} (\mathbf{N}_1^t \mathbf{C} \mathbf{N}_1) && (5)\\
&= \frac{\partial \mathbf{N}_1^t}{\partial \mathbf{x}} \mathbf{C}\mathbf{N}_1 + \mathbf{N}_1^t \frac{\partial \mathbf{C}}{\partial \mathbf{x}} \mathbf{N}_1 + \mathbf{N}_1^t \mathbf{C} \frac{\partial \mathbf{N}_1}{\partial \mathbf{x}} && (6)\\
&= \left(\frac{\partial \mathbf{N}_1}{\partial \alpha} \frac{\partial \alpha}{\partial \mathbf{x}} \right)^t \mathbf{C}\mathbf{N}_1 + \mathbf{N}_1^t \frac{\partial \mathbf{C}}{\partial \mathbf{x}} \mathbf{N}_1 + \mathbf{N}_1^t \mathbf{C} \left(\frac{\partial \mathbf{N}_1}{\partial \alpha} \frac{\partial \alpha}{\partial \mathbf{x}} \right) && (7)\\
&= \frac{\partial \alpha}{\partial \mathbf{x}}^t \mathbf{N}_2^t \mathbf{C}\mathbf{N}_1 + \mathbf{N}_1^t \frac{\partial \mathbf{C}}{\partial \mathbf{x}} \mathbf{N}_1 + \mathbf{N}_1^t \mathbf{C}\mathbf{N}_2 \frac{\partial \alpha}{\partial \mathbf{x}} && (8)\\
&= \mathbf{N}_1^t \frac{\partial \mathbf{C}}{\partial \mathbf{x}} \mathbf{N}_1 \ , && (9)
\end{aligned}$$

[1]the stretch in the thickness direction is inferred from in-plane stretches assuming constant volume, i.e., $\det \mathbf{C} = 1$.

where the last transformation follows from the fact that $\mathbf{C}\mathbf{N}_1 = \lambda_1\mathbf{N}_1$ and $\mathbf{N}_2^t\mathbf{N}_1 = 0$. It is evident from these expressions that the only derivatives required are those of $\mathbf{C}$, which are easily computed[2].

The second derivative follows as

$$\begin{aligned}
\frac{\partial^2\lambda_1}{\partial\mathbf{x}^2} &= \frac{\partial}{\partial\mathbf{x}}(\mathbf{N}_1^t\frac{\partial\mathbf{C}}{\partial\mathbf{x}}\mathbf{N}_1) && (10)\\
&= \frac{\partial\mathbf{N}_1^t}{\partial\mathbf{x}}\frac{\partial\mathbf{C}}{\partial\mathbf{x}}\mathbf{N}_1 + \mathbf{N}_1^t\frac{\partial^2\mathbf{C}}{\partial\mathbf{x}^2}\mathbf{N}_1 + \mathbf{N}_1^t\frac{\partial\mathbf{C}}{\partial\mathbf{x}}\frac{\partial\mathbf{N}_1^t}{\partial\mathbf{x}} && (11)\\
&= \left(\frac{\partial\mathbf{N}_1}{\partial\alpha}\frac{\partial\alpha}{\partial\mathbf{x}}\right)^t\frac{\partial\mathbf{C}}{\partial\mathbf{x}}\mathbf{N}_1 + \mathbf{N}_1^t\frac{\partial^2\mathbf{C}}{\partial\mathbf{x}^2}\mathbf{N}_1 + \mathbf{N}_1^t\frac{\partial\mathbf{C}}{\partial\mathbf{x}}\left(\frac{\partial\mathbf{N}_1}{\partial\alpha}\frac{\partial\alpha}{\partial\mathbf{x}}\right) && (12)\\
&= \frac{\partial\alpha}{\partial\mathbf{x}}^t\mathbf{N}_2\frac{\partial\mathbf{C}}{\partial\mathbf{x}}\mathbf{N}_1 + \mathbf{N}_1^t\frac{\partial^2\mathbf{C}}{\partial\mathbf{x}^2}\mathbf{N}_1 + \mathbf{N}_1^t\frac{\partial\mathbf{C}}{\partial\mathbf{x}}\mathbf{N}_2\frac{\partial\alpha}{\partial\mathbf{x}}\ . && (13)
\end{aligned}$$

The first and last term of (13) can be further simplified. Using the fact that $\mathbf{N}_1^t\mathbf{C}\mathbf{N}_2 = 0$ we have

$$\frac{\partial}{\partial\mathbf{x}}(\mathbf{N}_2^t\mathbf{C}\mathbf{N}_1) = \mathbf{N}_2^t\frac{\partial\mathbf{C}}{\partial\mathbf{x}}\mathbf{N}_1 + \frac{\partial\mathbf{N}_2}{\partial\mathbf{x}}^t\mathbf{C}\mathbf{N}_1 + \mathbf{N}_2^t\mathbf{C}\frac{\partial\mathbf{N}_1}{\partial\mathbf{x}} = 0\ , \tag{14}$$

and therefore

$$\begin{aligned}
\mathbf{N}_2^t\frac{\partial\mathbf{C}}{\partial\mathbf{x}}\mathbf{N}_1 &= -\left(\frac{\partial\mathbf{N}_2}{\partial\alpha}\frac{\partial\alpha}{\partial\mathbf{x}}\right)^t\mathbf{C}\mathbf{N}_1 - \mathbf{N}_2^t\mathbf{C}\left(\frac{\partial\mathbf{N}_1}{\partial\alpha}\frac{\partial\alpha}{\partial\mathbf{x}}\right) && (15)\\
&= \frac{\partial\alpha}{\partial\mathbf{x}}^t\mathbf{N}_1\mathbf{C}\mathbf{N}_1 - \mathbf{N}_2^t\mathbf{C}\mathbf{N}_2\frac{\partial\alpha}{\partial\mathbf{x}} && (16)\\
&= (\lambda_1 - \lambda_2)\frac{\partial\alpha}{\partial\mathbf{x}}^t\ . && (17)
\end{aligned}$$

Using (17) in (13), we have

$$\frac{\partial^2\lambda_1}{\partial\mathbf{x}^2} = 2(\lambda_1 - \lambda_2)\frac{\partial\alpha}{\partial\mathbf{x}}\frac{\partial\alpha}{\partial\mathbf{x}}^t + \mathbf{N}_1^t\frac{\partial^2\mathbf{C}}{\partial\mathbf{x}^2}\mathbf{N}_1\ . \tag{18}$$

Finally, by using (17) in (15) we obtain

$$\frac{\partial\alpha}{\partial\mathbf{x}}^t = \frac{1}{\lambda_1 - \lambda_2}\mathbf{N}_2^t\frac{\partial\mathbf{C}}{\partial\mathbf{x}}\mathbf{N}_1 \tag{19}$$

and can thus compute the second derivative of λ_1.

Remarks

It should be noted that (17) and (19) are not well-defined when $\lambda_1 = \lambda_2$, in which case the eigen-decomposition of $\mathbf{C}$ becomes non-unique anyways. For the application described in the main article, however, this problem does not arise since the case of $\lambda_1 = \lambda_2$ is handled by the original energy density

$$\psi(\lambda_1, \lambda_2) = \kappa\left(\lambda_1 + \lambda_2 + \frac{1}{\lambda_1\lambda_2} - 3\right) = \kappa\left(\mathrm{tr}(\mathbf{C}) + \frac{1}{\det\mathbf{C}} - 3\right)\ , \tag{20}$$

which does not require eigenvalues.

[2]Using Constant Strain Triangles for discretization, the deformation gradient $\mathbf{F}$ is a linear function of $\mathbf{x}$ such that $\mathbf{C}$ is quadratic in positions.

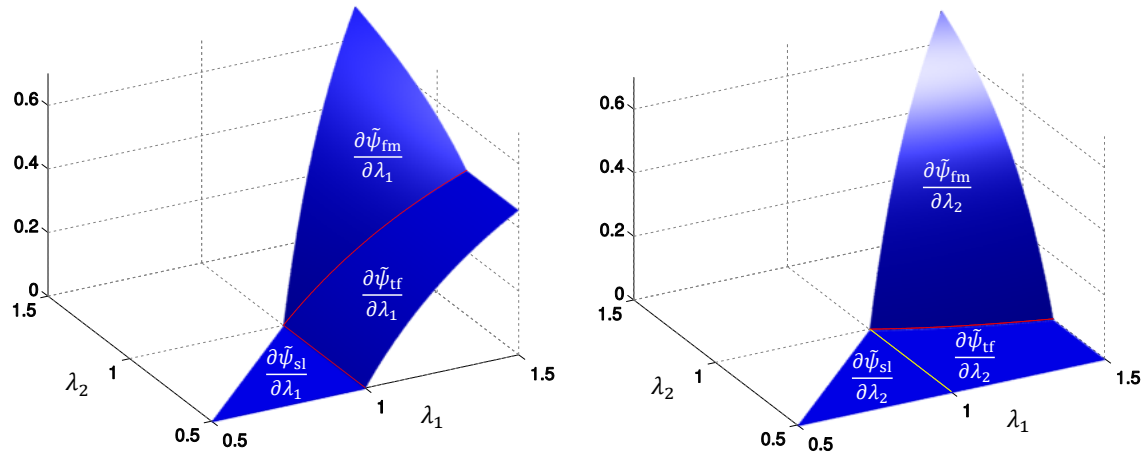

Figure 1: Landscape of the relaxed energy derivatives. Discontinuity lines (red) are clearly visible at the transition between the different regimes.

2 Smoothing the Modified Energy Density Derivatives

The original energy density defined by (20) leads to numerical problems when elements are compressed. Following tension field theory, the macroscopic behavior of wrinkled regions can be captured using a relaxed energy density formulation defined as

$$\tilde{\psi}(\lambda_1, \lambda_2) = \begin{cases} \tilde{\psi}_{\mathrm{sl}}(\lambda_1, \lambda_2) = 0 & \lambda_1 < 1, \lambda_2 < 1 \\ \tilde{\psi}_{\mathrm{tf}}(\lambda_1, \lambda_2) = \psi(\lambda_1, \tilde{\lambda}_2(\lambda_1)) & \lambda_1 \geq 1, \lambda_2 < \tilde{\lambda}_2(\lambda_1) \\ \tilde{\psi}_{\mathrm{fm}}(\lambda_1, \lambda_2) = \psi(\lambda_1, \lambda_2) & \lambda_1 \geq 1, \lambda_2 \geq \tilde{\lambda}_2(\lambda_1) \end{cases} , \tag{21}$$

where $\tilde{\lambda}_2$ is the energetic minimum of λ_2,

$$\tilde{\lambda}_2(\lambda_1) = \underset{\lambda_2}{\operatorname{argmin}}\ \psi(\lambda_1, \lambda_2) = \frac{1}{\sqrt{\lambda_1}} . \tag{22}$$

This relaxed energy density can then be explicitly written as

$$\tilde{\psi}(\lambda_1, \lambda_2) = \begin{cases} \tilde{\psi}_{\mathrm{sl}}(\lambda_1, \lambda_2) = 0 & \lambda_1 < 1, \lambda_2 < 1 \\ \tilde{\psi}_{\mathrm{tf}}(\lambda_1, \lambda_2) = \kappa \left(\lambda_1 + \frac{2}{\sqrt{\lambda_1}} - 3 \right) & \lambda_1 \geq 1, \lambda_2 < \frac{1}{\sqrt{\lambda_1}} \\ \tilde{\psi}_{\mathrm{fm}}(\lambda_1, \lambda_2) = \kappa \left(\lambda_1 + \lambda_2 + \frac{1}{\lambda_1 \lambda_2} - 3 \right) & \lambda_1 \geq 1, \lambda_2 \geq \frac{1}{\sqrt{\lambda_1}} \end{cases} . \tag{23}$$

The energy density $\tilde{\psi}$ is convex but only C^1 continuous and the discontinuities in the second derivatives are clearly visible in Fig. 1.

While we did not experience any adverse effects when solving for equilibrium states, the discontinuous force derivatives pose a significant problem for optimization since the formulation of the constraints directly depends on these forces. Instead of turning toward sophisticated methods for

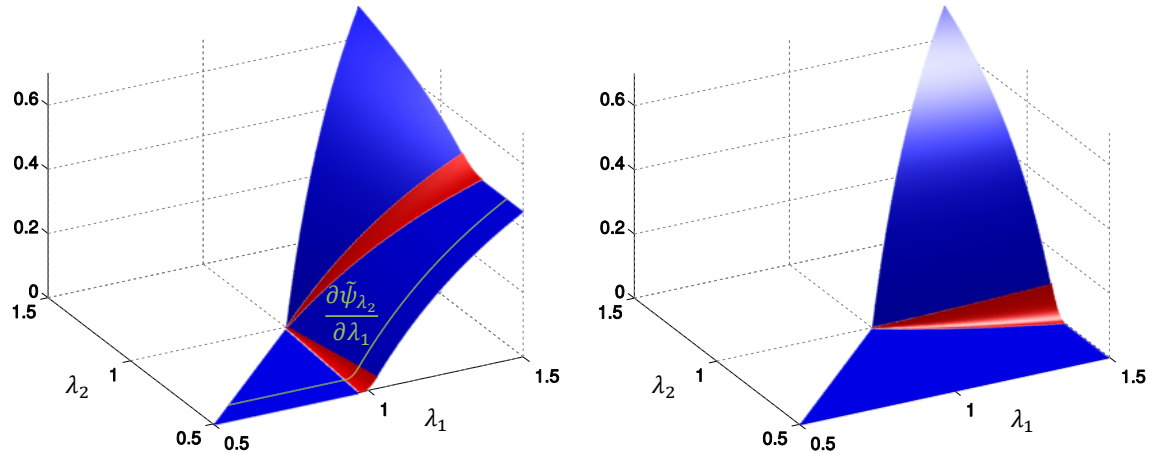

Figure 2: Landscape of the relaxed energy derivatives after smoothing. Discontinuities have been replaced by interpolating transition zones (red) using quadratic functions.

nonsmooth optimization, we preferred to smooth the transitions between the different regimes using quadratic interpolation as illustrated on Fig. 2 and detailed below.

We start by considering the simple one-dimensional problem shown in Fig. 3, where c is a constant, $\epsilon > 0$ and g_1 and g_2 are two scalar functions defined by

$$g_1 : x \mapsto 0 \quad \text{and} \quad g_2 : x \mapsto cx \ . \tag{24}$$

It is possible to define a quadratic transition function h over $[-\epsilon, +\epsilon]$ interpolating g_1 and g_2 and their first derivatives such that

$$\begin{array}{rcl} g_1(-\epsilon) & = & h(-\epsilon) \ , \\ g_1'(-\epsilon) & = & h'(-\epsilon) \ , \\ g_2(\epsilon) & = & h(\epsilon) \ , \\ g_2'(\epsilon) & = & h'(\epsilon) \ . \end{array} \tag{25}$$

This function is defined by

$$h : x \mapsto \frac{c}{4\epsilon}x^2 + \frac{c}{2}x + \frac{c\epsilon}{4}, \quad x \in [-\epsilon, \epsilon]. \tag{26}$$

This result can be applied with little modification to the slightly more general case where g_1 and g_2 are defined by

$$g_1 : x \mapsto a \quad \text{and} \quad g_2 : x \mapsto cx + d \ , \tag{27}$$

with a, d and $c \neq 0$ given constant parameters.

Let $x_I = \frac{a-d}{c}$ denote the abscissa of the intersection between the graphs of g_1 and g_2. We can now define the transition function h over $[x_I - \epsilon, x_I + \epsilon]$ which interpolates g_1 and g_2 at $\{x_I - \epsilon\}$ and $\{x_I + \epsilon\}$ by

$$h : x \mapsto \frac{c}{4\epsilon}t^2 + \frac{c}{2}t + \frac{c\epsilon}{4} + a, \ t = x - x_I, \ x \in [x_I - \epsilon, x_I + \epsilon]. \tag{28}$$

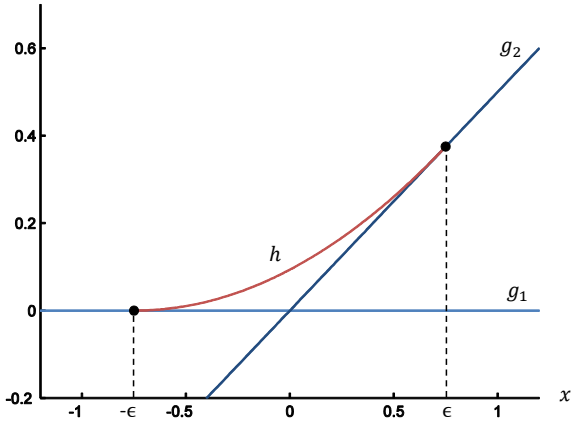

Figure 3: Illustration for the construction of a transition function h interpolating the two scalar functions g_1 and g_2 of equation (24) and their first derivatives.

We can now focus on our original problem, i.e. the smoothing of the derivatives of the relaxed energy $\tilde{\psi}$. The discontinuities in its second derivatives appears between $\frac{\partial \tilde{\psi}_{\mathrm{sl}}}{\partial \lambda_1}$ and $\frac{\partial \tilde{\psi}_{\mathrm{tf}}}{\partial \lambda_1}$, $\frac{\partial \tilde{\psi}_{\mathrm{tf}}}{\partial \lambda_1}$ and $\frac{\partial \tilde{\psi}_{\mathrm{fm}}}{\partial \lambda_1}$, and $\frac{\partial \tilde{\psi}_{\mathrm{tf}}}{\partial \lambda_2}$ and $\frac{\partial \tilde{\psi}_{\mathrm{fm}}}{\partial \lambda_2}$. In what follows, we detail the construction of the transition function between $\frac{\partial \tilde{\psi}_{\mathrm{sl}}}{\partial \lambda_1}$ and $\frac{\partial \tilde{\psi}_{\mathrm{tf}}}{\partial \lambda_1}$. The approach can be directly adapted to smooth out the two other discontinuity lines.

For every λ_2, we keep λ_2 fixed and smooth the first derivative of the univariate function $\tilde{\psi}_{\lambda_2} : \lambda_1 \mapsto \tilde{\psi}(\lambda_1, \lambda_2)$. In order to make the size of the transition interval converge to 0 as λ_1 and λ_2 converge to $(1, 1)$, we start by defining its upper bound u by

$$u = 1 + \delta(1 - \lambda_2) \ , \tag{29}$$

where δ is a given parameter.

We finally define the function interpolating $\tilde{\psi}_{\lambda_2}$ using the previous result (equation 28) with

$$\begin{aligned} a &= \frac{\partial \tilde{\psi}_{\mathrm{sl}}}{\partial \lambda_1}(1, \lambda_2) \ , \\ c &= \frac{\partial^2 \tilde{\psi}_{\mathrm{sl}}}{\partial \lambda_1^2}(u, \lambda_2) \ , \\ \epsilon &= \frac{1}{c} \frac{\partial \tilde{\psi}_{\mathrm{tf}}}{\partial \lambda_1}(u, \lambda_2) \ , \\ x_I &= u - \epsilon, \end{aligned} \tag{30}$$

as illustrated in Fig. 4.

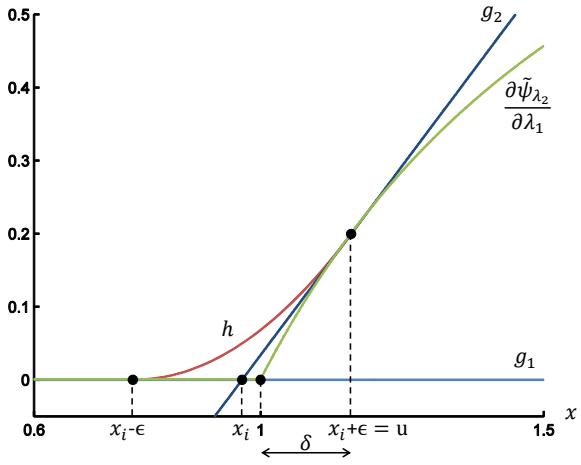

Figure 4: Illustration for the construction of a transition function h smoothing the first derivative of the univariate function $\tilde{\psi}_{\lambda_2}$.