Example-based Structure Synthesis: Derivations

paper 1054

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1 Deriving the Discrete Similarity Measure

For the derivations in this document, we denote dot product of two vectors with $\mathbf{xy}$, and the squared norm of a vector with $\mathbf{x}^2$ for brevity. If we substitute the sums of Gaussians in Equation 2 of the paper we obtain:

$$
\begin{aligned}
&S(\mathbf{f}(\mathbf{x}), \mathbf{e}(\mathbf{m}(\mathbf{x}))) = \\
&= \int \left| \sum_i \mathbf{a}_i g(\mathbf{x} + \mathbf{s} - \mathbf{x}_i, \sigma) - \sum_i \mathbf{b}_i g(\mathbf{m}(\mathbf{x}) + \mathbf{s} - \mathbf{e}_i, \sigma) \right|^2 g(\mathbf{s}, \delta) \mathbf{ds} \qquad (1) \\
&= \int \sum_{ij} (\mathbf{a}_i \mathbf{a}_j) g(\mathbf{x} + \mathbf{s} - \mathbf{x}_i, \sigma) g(\mathbf{x} + \mathbf{s} - \mathbf{x}_j, \sigma) g(\mathbf{s}, \delta) \\
&\quad - 2 \sum_{ij} (\mathbf{a}_i \mathbf{b}_j) g(\mathbf{x} + \mathbf{s} - \mathbf{x}_i, \sigma) g(\mathbf{m}(\mathbf{x}) + \mathbf{s} - \mathbf{e}_j, \sigma) g(\mathbf{s}, \delta) \\
&\quad + \sum_{ij} (\mathbf{b}_i \mathbf{b}_j) g(\mathbf{m}(\mathbf{x}) + \mathbf{s} - \mathbf{e}_i, \sigma) g(\mathbf{m}(\mathbf{x}) + \mathbf{s} - \mathbf{e}_j, \sigma) g(\mathbf{s}, \delta) \mathbf{ds}
\end{aligned}
\qquad (2)
$$

The products of Gaussians lead to the following form:

$$
\begin{aligned}
&S(\mathbf{f}(\mathbf{x}), \mathbf{e}(\mathbf{m}(\mathbf{x}))) = \\
&= \int \sum_{ij} (\mathbf{a}_i \mathbf{a}_j) \exp\left(\frac{-2 - \frac{\sigma}{\delta}^2}{\sigma^2} \mathbf{s}^2 + \frac{-4\mathbf{x} + 2\mathbf{x_i} + 2\mathbf{x_j}}{\sigma^2} \mathbf{s} + \frac{-2\mathbf{x}^2 - \mathbf{x_i}^2 - \mathbf{x_j}^2 + 2\mathbf{x}\mathbf{x_i} + 2\mathbf{x}\mathbf{x_j}}{\sigma^2} \right) \\
&\quad - 2 \sum_{ij} (\mathbf{a}_i \mathbf{b}_j) \exp\left(\frac{-2 - \frac{\sigma}{\delta}^2}{\sigma^2} \mathbf{s}^2 + \frac{-2\mathbf{x} - 2\mathbf{m}(\mathbf{x}) + 2\mathbf{x_i} + 2\mathbf{e_j}}{\sigma^2} \mathbf{s} + \frac{-\mathbf{x}^2 - \mathbf{m}(\mathbf{x})^2 - \mathbf{x_i}^2 - \mathbf{e_j}^2 + 2\mathbf{x}\mathbf{x_i} + 2\mathbf{x}\mathbf{e_j}}{\sigma^2} \right) \\
&\quad + \sum_{ij} (\mathbf{b}_i \mathbf{b}_j) \exp\left(\frac{-2 - \frac{\sigma}{\delta}^2}{\sigma^2} \mathbf{s}^2 + \frac{-4\mathbf{m}(\mathbf{x}) + 2\mathbf{e_i} + 2\mathbf{e_j}}{\sigma^2} \mathbf{s} + \frac{-2\mathbf{m}(\mathbf{x})^2 - \mathbf{e_i}^2 - \mathbf{e_j}^2 + 2\mathbf{m}(\mathbf{x})\mathbf{e_i} + 2\mathbf{m}(\mathbf{x})\mathbf{e_j}}{\sigma^2} \right) \mathbf{ds}
\end{aligned}
\qquad (3)
$$

We now use the integral form of a Gaussian function:

$$
\int k \exp(-f\mathbf{x}^2 + g\mathbf{x} + h) d\mathbf{x} = k\sqrt{\frac{\pi}{f}} \exp(\frac{g^2}{4f} + h) \qquad (4)
$$

Thus:

$$
\begin{aligned}
&S(\mathbf{f}(\mathbf{x}), \mathbf{e}(\mathbf{m}(\mathbf{x}))) = \\
&= c'' \Bigg[\sum_{ij} (\mathbf{a}_i \mathbf{a}_j) \exp\left(\frac{-(\mathbf{x_i} - \mathbf{x_j})^2 - (\frac{\sigma}{\delta})^2 (\mathbf{x} - \mathbf{x_i})^2 - (\frac{\sigma}{\delta})^2 (\mathbf{x} - \mathbf{x_j})^2}{2\sigma^2 + \frac{\sigma^4}{\delta^2}} \right) \\
&\quad - 2 \sum_{ij} (\mathbf{a}_i \mathbf{b}_j) \exp\left(\frac{-((\mathbf{m}(\mathbf{x}) - \mathbf{e_i}) - (\mathbf{x} - \mathbf{x_j}))^2 - (\frac{\sigma}{\delta})^2 (\mathbf{m}(\mathbf{x}) - \mathbf{e_i})^2 - (\frac{\sigma}{\delta})^2 (\mathbf{x} - \mathbf{x_j})^2}{2\sigma^2 + \frac{\sigma^4}{\delta^2}} \right) \\
&\quad + \sum_{ij} (\mathbf{b}_i \mathbf{b}_j) \exp\left(\frac{-(\mathbf{e_i} - \mathbf{e_j})^2 - (\frac{\sigma}{\delta})^2 (\mathbf{m}(\mathbf{x}) - \mathbf{e_i})^2 - (\frac{\sigma}{\delta})^2 (\mathbf{m}(\mathbf{x}) - \mathbf{e_j})^2}{2\sigma^2 + \frac{\sigma^4}{\delta^2}} \right) \Bigg]
\end{aligned}
\qquad (5)
$$

where $c'' = \sqrt{\frac{\pi\sigma^2}{2+(\frac{\sigma}{\delta})^2}}$

Finally, the following form is reached:

$$
\begin{aligned}
&S(\mathbf{f}(\mathbf{x}), \mathbf{e}(\mathbf{m}(\mathbf{x}))) = \\
=&c'\Bigg(\sum_{ij}(\mathbf{a}_i\mathbf{a}_j)g(\mathbf{x_i}-\mathbf{x_j}, \sigma c)g(\mathbf{x}-\mathbf{x_i}, \delta c)g(\mathbf{x}-\mathbf{x_j}, \delta c) \\
&-2\sum_{ij}(\mathbf{a}_i\mathbf{b}_j)g((\mathbf{m}(\mathbf{x})-\mathbf{e_i})-(\mathbf{x}-\mathbf{x_j}), \sigma c)g(\mathbf{m}(\mathbf{x})-\mathbf{e_i}, \delta c)g(\mathbf{x}-\mathbf{x_j}, \delta c) \\
&+\sum_{ij}(\mathbf{b}_i\mathbf{b}_j)g(\mathbf{e_i}-\mathbf{e_j}, \sigma c)g(\mathbf{m}(\mathbf{x})-\mathbf{e_i}, \delta c)g(\mathbf{m}(\mathbf{x})-\mathbf{e_j}, \delta c)\Bigg)
\end{aligned}
\tag{6}
$$

where $c = \sqrt{2+(\frac{\sigma}{\delta})^2}$ and $c' = \frac{\sqrt{\pi\sigma^2}}{c}$.

While Equation 6 of the supplementary material can be used as the similarity measure and its gradients can be computed, we propose a slight adaptation which allows us to formulate the discrete similarity measure in a more compact form, and results in a negligable change in the similarity. Instead of using a Gaussian for the window function, one can use a box function with size δ. Then, Equation 1 of the supplementary material can be approximated with:

$$S(\mathbf{f}(\mathbf{x}), \mathbf{e}(\mathbf{m}(\mathbf{x}))) = \int \left|\sum_i \mathbf{a}_i g(\mathbf{x}+\mathbf{s}-\mathbf{x}_i, \sigma) - \sum_i \mathbf{b}_i g(\mathbf{m}(\mathbf{x})+\mathbf{s}-\mathbf{e}_i, \sigma)\right|^2 \mathbf{ds}, \tag{7}$$

where only the $\mathbf{x_i}$ closer than δ to $\mathbf{x}$ and the $\mathbf{e_i}$ closer than δ to $\mathbf{m}(\mathbf{x})$ are considered. Following the steps above, the following final form is reached:

$$
\begin{aligned}
&S(\mathbf{f}(\mathbf{x}), \mathbf{e}(\mathbf{m}(\mathbf{x}))) = \\
&\frac{\sqrt{\pi\sigma^2}}{2}\Bigg(\sum_{ij}(\mathbf{a}_i\mathbf{a}_j)g(\mathbf{x_i}-\mathbf{x_j}, \sqrt{2}\sigma) - 2\sum_{ij}(\mathbf{a}_i\mathbf{b}_j)g((\mathbf{m}(\mathbf{x})-\mathbf{e_i})-(\mathbf{x}-\mathbf{x_j}), \sqrt{2}\sigma) + \sum_{ij}(\mathbf{b}_i\mathbf{b}_j)g(\mathbf{e_i}-\mathbf{e_j}, \sqrt{2}\sigma)\Bigg)
\end{aligned}
\tag{8}
$$

2 Computing the Gradients

$$\frac{\partial T}{\partial \mathbf{m_k}} = -\frac{\sqrt{2\pi}}{\sigma}\sum_{ij}(\mathbf{a}_i\mathbf{b}_j)g((\mathbf{m_k}-\mathbf{e_i})-(\mathbf{q_k}-\mathbf{x_j}), \sqrt{2}\sigma)((\mathbf{m_k}-\mathbf{e_i})-(\mathbf{q_k}-\mathbf{x_j})) \tag{9}$$

$$
\begin{aligned}
\frac{\partial T}{\partial \mathbf{x_i}} =& -\frac{\sqrt{2\pi}}{\sigma}\sum_k\Bigg(\sum_j(\mathbf{a}_i\mathbf{b}_j)g((\mathbf{m_k}-\mathbf{e_j})-(\mathbf{q_k}-\mathbf{x_i}), \sqrt{2}\sigma)((\mathbf{m_k}-\mathbf{e_j})-(\mathbf{q_k}-\mathbf{x_i})) \\
&+\sum_j(\mathbf{a}_i\mathbf{a}_j)g(\mathbf{x_j}-\mathbf{x_i}, \sqrt{2}\sigma)(\mathbf{x_j}-\mathbf{x_i})\Bigg)
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
\frac{\partial T}{\partial \mathbf{a_i}} =& \frac{\sqrt{\pi\sigma^2}}{2}\sum_k\Bigg(\sum_j \mathbf{a}_j g(\mathbf{x_i}-\mathbf{x_j}, \sqrt{2}\sigma) \\
&-2\sum_j \mathbf{b}_j g((\mathbf{m_k}-\mathbf{e_i})-(\mathbf{x_k}-\mathbf{x_j}), \sqrt{2}\sigma)\Bigg)
\end{aligned}
\tag{11}
$$

where only input points $\mathbf{e_i}$ and output points $\mathbf{x_i}$ close enough to $\mathbf{m_k}$ and $\mathbf{q_k}$ for the current k are considered.